

The Cambridge controversies in the theory of capital: contributions from the complex plane

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Abstract

A controversy in capital theory concerns reswitching. When two production techniques are compared, reswitching occurs when one technique is cheapest at low interest rates, switches to being more expensive at higher rates, and then reswitches to being cheapest at yet higher rates. Some believe this inconsistency undermines neoclassical economics. The time-value-of-money (TVM) equation is at the core of the puzzle. The equation is a polynomial having n roots, implying n interest rates. In most analyses, including reswitching, one interest rate is used and the remaining rates are ignored. This analysis demonstrates that every TVM equation has a ‘dual’ form employing all interest rates. The dual of the reswitching equation explains the puzzle.

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This article explains a longstanding puzzle in economic theory: reswitching. The explanation is provided by multiple-interest-rate analysis.

Reswitching is part of the Cambridge controversies in capital theory. The controversies surfaced at the beginning of the twentieth century, intensified into the ‘Cambridge Controversies’ during the 1960s, and have simmered since. A high point of the debate is the 1966 symposium on reswitching in the *Quarterly Journal of Economics* containing articles by Bruno *et al.* (1966), Garegnani (1966), Levhari and Samuelson (1966), Morishima (1966), Pasinetti (1966), and Samuelson (1966). Harcourt (1972) is a survey of the controversies; Cohen and Harcourt (2003) is a recent review.

When two techniques of production are compared, reswitching is the possibility one technique is cheapest at a low interest rate, switches to being more expensive at a higher rate, and reswitches to being cheapest again at even higher rates. For some, this inconsistency undermines the foundations of neoclassical economic theory.

Samuelson (1966) expressed concern about reswitching thus:

‘The phenomenon of switching back at a very low interest rate to a set of techniques that had seemed viable only at a very high interest rate involves more than esoteric technicalities. It shows the simple tale told by Jevons, Bohm-Bawerk, Wicksell, and other neoclassical writers ... cannot be universally valid.’

Solow (1963) was of the opinion that ‘... *when a theoretical question remains debatable after 80 years there is a presumption that the question is badly posed -- or very deep indeed. I believe that the question of the measurement of 'capital' is badly posed ...*’

Forty years later, with no resolution in sight, Cohen and Harcourt (2003) were able to write ‘*Solow defended the ‘badly posed’ answer, but we believe that the questions at issue in the recurring capital controversies are ‘very deep indeed.’*’

This article contains a new approach to the puzzle. Reswitching analysis employs the time value of money (TVM) equation, a key equation in economics and finance. The TVM equation takes the form of an n^{th} order polynomial having n roots, every root containing an interest rate. In most economic and financial analyses, including reswitching, it is normal to employ only one root, namely the root implying a positive, real-valued interest

rate. The $(n-1)$ unorthodox roots are mostly complex numbers or negative real numbers, and are usually ignored; when not ignored they are seen as problems. An early example of the latter is Wright (1936); a recent example is Magni (2010); both examples are from the capital budgeting literature.

We demonstrate that every conventional TVM equation has a dual expression. The principal feature of a dual expression is that it contains every interest rate solving its conventional counterpart. The dual expression for the reswitching equation explains the reswitching puzzle.

The Sraffa-Samuelson model of reswitching

Samuelson (1966) contains a small, numerical model capturing the phenomenon of reswitching described in Sraffa (1960).¹ Samuelson's model is often quoted; see Cohen and Harcourt (2003) and Harcourt (2006). The model is simple and apt; therefore it is employed in this article.

The model assumes the value (or cost) of capital today is the accumulated value of past labor inputs, i.e. capital is stored (or dated or frozen) labor. Values of past labor inputs are compounded to the present value of capital. Eq. (1) shows labor inputs at three moments in the past (L_i from $i = 1$ to 3) compounded at a rate of interest (r) to today's capital value (C).²

$$C = L_1(1+r) + L_2(1+r)^2 + L_3(1+r)^3 \quad (1)$$

Table 1 shows the labor inputs assumed by Samuelson (1966) for two production techniques, A and B.

[Table 1 about here.]

The numbers in Table 1 applied to Eq. (1) result in the following equations for the two techniques.

$$C_A = 7(1+r)^2 \quad (2)$$

$$C_B = 6(1+r) + 2(1+r)^3 \quad (3)$$

At rates of interest between 0% and 50% technique B is more expensive than technique A, then switches to being cheaper at rates between 50% and 100%, and finally reswitches to being more expensive again at rates above 100%. Figure 1 illustrates the situation.

[Figure 1 about here.]

Reswitching is a puzzling feature of the relationship between the value of capital and the rate of interest. It is argued here that the relationship between C and r is subtler than it appears, the subtlety arising from the functional form of Eq. (1). The equation is a polynomial of order three; therefore for each capital value C there are three values of $(1+r)$ solving the equation. These multiple interest rates are examined next.

Multiple interest rates

In general, a TVM polynomial of order n has n solutions for $(1+r)$, i.e. $(1+r_j)$ from $j = 1$ to n . In the current analysis, the root $(1+r_1)$ is designated the orthodox root, and the remaining $(n-1)$ roots from $j = 2$ to n are labeled ‘unorthodox.’ The unorthodox roots are so-called because they include negative real numbers and complex numbers of the form $a + bi$ where i is the imaginary number, $\sqrt{-1}$.

The existence of three solutions for $(1+r)$ in Eq. (1) implies three versions of Eq. (1). The three equations hold true simultaneously.

$$C = L_1(1+r_1) + L_2(1+r_1)^2 + L_3(1+r_1)^3 \quad (1a)$$

$$C = L_1(1+r_2) + L_2(1+r_2)^2 + L_3(1+r_2)^3 \quad (1b)$$

$$C = L_1(1+r_3) + L_2(1+r_3)^2 + L_3(1+r_3)^3 \quad (1c)$$

Given values for the labor inputs L_i and capital value C , then the three interest rates r_j are determined. Alternatively, given values for the labor inputs L_i and the orthodox rate of interest r_1 , then capital value C and the interest rates r_2 and r_3 are determined.

The particular values for L_i in Table 1 mean that technique A is described by two equations, each having its own rate of interest. Equations (4) and (5) employ rates r_{A1} and r_{A2} respectively.

$$C_A = 7(1+r_{A1})^2 \quad (4)$$

$$C_A = 7(1+r_{A2})^2 \quad (5)$$

For example, if the interest rate r_{A1} is 10%, i.e. $(1+r_{A1}) = 1.1$, then C_A is 8.470. Given $C_A = 8.470$, the root $(1+r_{A2}) = -1.1$ is also a solution, i.e. $r_{A2} = -2.10$ (-210%).

Similarly, technique B is described by three equations, each having its own rate of interest. Equations (6), (7), and (8) employ rates r_{B1} , r_{B2} and r_{B3} respectively.

$$C_B = 6(1+r_{B1}) + 2(1+r_{B1})^3 \quad (6)$$

$$C_B = 6(1+r_{B2}) + 2(1+r_{B2})^3 \quad (7)$$

$$C_B = 6(1+r_{B3}) + 2(1+r_{B3})^3 \quad (8)$$

If the orthodox interest rate r_{B1} is 10%, i.e. $(1+r_{B1}) = 1.1$, then C_B is 9.262. Given $C_B = 9.262$, there are two complex-valued solutions: $(1+r_{B2}) = -0.55+0.97674i$ and $(1+r_{B3}) = -0.55-0.97674i$. This situation generalizes – see Appendix A(i).

Mathematically each solution for the interest rate is as valid as any other. Conventional opinion in economics and finance, however, has not viewed all solutions as equally valid. The unorthodox solutions to the time value of money equation have been ignored for a long time. The probable reason for the neglect is that the negative and complex solutions are not obvious candidates for inclusion in a practical economic theory; some influential economists have been of the opinion that such solutions have neither use nor meaning.

Here is an early statement by Boulding (1936) made during a debate about capital budgeting:

'Now it is true that an equation of the n th degree has n roots of one sort or another ... Nevertheless, in the type of payments series with which we are most likely to be concerned, it is extremely probable that all but one of these roots will be either negative or imaginary [complex], in which case they will have no economic significance.'

A later example of a similar assertion comes from Soper (1959):

‘... there must be n roots, and hence n possible values of r . Some of the roots can be ignored as irrelevant; those which are less than zero or complex.’

This conventional view, that negative and complex roots have neither use nor meaning, is well entrenched. For example, when asked to calculate the yield or IRR from a sequence of cash flows, financial calculators and spreadsheets report only the orthodox value r_1 because that is what they are programmed to do. The conventional view is written into the software.³

There is a small, multiple-interest-rate literature in the context of the Cambridge capital controversies. Bharadwaj (1970) mentions all roots but explicitly excludes the complex solutions, arguing, like Boulding (1936) and Soper (1959), that they are not relevant. Harcourt (1972) notes that Bruno *et al.* (1966) mention multiple roots. Hagemann and Kurz (1976) discuss multiple roots at length, concluding that *‘... a close connection between the multiplicity of the internal rate of return and the reswitching of techniques ... does not exist.’* However, the latter two works restrict their analyses to multiple *real-valued* roots; they do not consider all possible roots, including the complex.

The unorthodox solutions are discussed in recent research, mostly in the field of capital budgeting. There are varied opinions about such solutions. Hartman and Schafrick (2004) and Magni (2010) mention the complex solutions but maintain the conventional view and find (different) ways to exclude them.

Hazen (2003) and Pierru (2010) are positive about the unorthodox solutions, finding (different) ways to employ them in the traditional manner, i.e. singly, as rates of return measuring the worth of an investment.

Dorfman (1981) is a seminal article employing the multiple solutions, not singly but in combination, as components in the formula for another economic concept; Osborne (2010) also takes this all-rates-together approach, as does the current analysis.

In order to explore the role played by every possible solution for the interest rate, an interim mathematical result is needed, a result transforming the conventional capital value equation (1) into its dual expression in which all interest rates are visible, functional and meaningful. This mathematical result is presented next.

The fundamental theorem of algebra, factorization, and the dual expression for capital value

Aleksandrov *et al.* (1969) summarize a well-known result about factorization of polynomials:

‘If we accept without proof the so-called fundamental theorem of algebra that every equation $f(x)=0$, where $f(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_n$ is a polynomial in x of given degree n and the coefficients a_1, a_2, \dots, a_n are given real or complex numbers, has at least one real or complex root, ... then it is easy to show that the polynomial $f(x)$ can be represented (and in only one way) as a product of first-degree factors $f(x) = (x-a)(x-b)\dots(x-l)$ where a, b, \dots, l are real or complex numbers.’

Aleksandrov *et al.* (1969, Vol.1, pp. 271-272)

These results, the fundamental theorem of algebra and polynomial factorization, imply that the conventional equation (1), in which capital value is a function of three parameters and the single orthodox root, can be transformed into a dual equation, in which capital value is a function of three parameters and *every* root of Eq. (1).

The capital value equation (1) is rearranged and factorized along the lines described above; the result is Eq. (9) in which $[(1+r)-(1+r_j)]$ is the j^{th} factor, $(1+r_j)$ is the j^{th} root, and r_j is the interest rate implied by the j^{th} root.

$$L_3(1+r)^3 + L_2(1+r)^2 + L_1(1+r) - C = L_3[(1+r)-(1+r_1)][(1+r)-(1+r_2)][(1+r)-(1+r_3)] \quad (9)$$

Theoretically, the variable $(1+r)$ in Eq. (9) can take any value; it can roam over the entire complex plane. Some values of $(1+r)$ are more economically interesting than others, for example, when $(1+r)$ takes the value of the orthodox root $(1+r_l)$, the right-hand side of (9) goes to zero, and Eq. (9) reduces to Eq. (1a), the conventional capital value equation. Another salient value for $(1+r)$ is unity, i.e. r is zero. When r is zero Eq. (9) reduces to Eq. (10) in which all values of r_j coexist.

$$\sum_{i=1}^3 L_i - C = L_3 \prod_{j=1}^3 (-r_j) \quad (10)$$

Eq. (10) rearranges into Eq. (11), which is the desired dual expression for the capital value equation, Eq. (1).

$$C = \sum_{i=1}^3 L_i - L_3 \prod_{j=1}^3 (-r_j) \quad (11)$$

This result generalizes to the n^{th} order equation; see Appendix A(ii).

Observations on the dual equation

The first observation to make about Eq. (11) is that it employs every interest rate r_j implied by the three roots $(1+r_j)$ solving Eq. (1). All information from the three equations (1a) thru (1c) is compressed into a single equation. The very existence of the dual Eq. (11) means that every interest rate has use.⁴

The second observation to make about Eq. (11) concerns the meaning of the product on the right-hand side. The conventional interest rate r_1 is understood already. However, the two, unorthodox interest rates r_2 and r_3 require explanation. It is stated here, and proved in Appendix B, that the product of the two, unorthodox interest rates is equal to the number of times the orthodox interest rate r_1 is applied to an initially invested unit of labor during amortization of the conventional equation (1a). This result means the unorthodox product has meaning in addition to use, therefore it is given its own label X_r . Eq. (11) becomes Eq. (12).

$$C = \sum_{i=1}^3 L_i - L_3 X_r (-r_1) \quad (12)$$

A third observation about the dual expression is that, in order to connect with economic reality, the orthodox interest rate r_1 is assumed to be a small, positive number. This assumption is built into every analysis of reswitching known to the authors. Under the assumption, Eq. (12) becomes Eq. (13). It should be noted, however, that the assumption is not mathematically necessary; the analysis works if the possibility of a negative, orthodox interest rate is retained.

$$C = \sum_{i=1}^3 L_i + L_3 X_r r_1 \quad (13)$$

Eq. (13) states that today's capital value (C) is equal to the uncompounded sum of the labor inputs (L_i) plus a product. The product is the initial labor input (L_3) multiplied by the

number of times the input is marked up during the amortization process (X_r), multiplied by the mark-up itself (r_I). The equation acts like a prism, splitting capital value (C) into the contribution from the uncompounded inputs (sum of L_i) and the contribution from the compounding process ($L_3 X_r r_I$).

A fourth observation about the dual equation is that all three interest rates hold true simultaneously – choose one, any one will do, and the other two are determined – therefore the independent variable is not r_I alone. The mapping from interest rate to capital value is not from r_I to C but from the composite variable $X_r r_I$ to C .⁵ Capital value C depends not only on the mark-up r_I , but also on the number of times the mark-up is applied, X_r .

An analogy is that the size of Lego brick comprising a Lego building is not, by itself, a good indicator of the building's size; the number of bricks comprising the building is as relevant to building size as size of brick.

This last observation is important. Although the dual Eq. (13) has a different appearance from the conventional capital value equation (1), the two equations perform similarly in that a given orthodox interest rate r_I implies the same value for C . The interpretation of the dual equation, however, is very different from the interpretation of the conventional equation: the relationship in Eq. (13) between the composite variable $X_r r_I$ and capital value C is linear in the parameters (labor inputs), therefore switching between two techniques can occur, but reswitching cannot. Two straight lines can cross once, but not twice.

Analyzing the dual equations to the Sraffa-Samuelson model

Eq. (13) applied to the two techniques A and B in the Sraffa-Samuelson model results in Eq. (14) and Eq. (15).

$$C_A = \sum_{i=1}^2 L_i - L_2 \prod_{j=1}^2 (-r_{Aj}) \quad (14)$$

$$C_B = \sum_{i=1}^3 L_i - L_3 \prod_{j=1}^3 (-r_{Bj}) \quad (15)$$

When the parameter values assumed by Samuelson are inserted into Eq. (14) and Eq. (15) the results are equations Eq. (16) and Eq. (17).

$$C_A = 7 - 7(-r_{A1})(-r_{A2}) \quad (16)$$

$$C_B = 8 - 2(-r_{B1})|r_{B2}||r_{B3}| \quad (17)$$

Tables 2a and 2b contain values for the roots $(1+r_{Aj})$ and $(1+r_{Bj})$ and the implied values of r_{Aj} and r_{Bj} when the orthodox rates are arbitrarily assigned the value of 10%, i.e. $(1+r_{AI}) = (1+r_{BI}) = 1.1$.

[Tables 2a and 2b about here.]

Values for the interest rates in Tables 2a and 2b inserted into equations (16) and (17) result in equations (18) and (19).

$$8.470 = 7 - 7(-0.1)(2.1) \quad (18)$$

$$9.262 = 8 - 2(-0.1)(2.51197)(2.51197) \quad (19)$$

Eq. (18) shows that the interest rate $r_{AI} = 0.1$ is applied 2.1 times to an initially invested unit of labor during the amortization process. Similarly, Eq. (19) shows that the interest rate $r_{BI} = 0.1$ is applied $(2.51197)^2 = 6.31$ times to an initially invested unit of labor during amortization. This analysis is confirmed by examining the amortization schedules for the two techniques – see Tables 3 and 4.

[Tables 3 and 4 about here.]

Figure 2 plots the result of a procedure. The procedure begins by entering values for r_{AI} and r_{BI} into equations (2) and (3) to obtain values for C_A and C_B ; this process then reverses; the values for C_A and C_B go back into equations (2) and (3) to solve for all values of r_j contributing to capital value. The resulting data is arranged according to the dual equations (16) and (17) and plotted in Figure 2. The variable on the horizontal axis is the product of every possible interest rate rather than the orthodox rate alone, i.e. the independent variable is the composite entity $X_r r_I$. The figure demonstrates switching but not reswitching.

[Figure 2 about here.]

Conclusion

In this article Sraffa's reswitching puzzle is re-examined in the context of the Samuelson model. The analysis focuses on the dual expression to the capital value equation rather than the capital value equation itself. The dual expression contains every possible interest rate solving the conventional equation. The rates are considered together, as a product, the orthodox rate acting as unit of value, and the product of its companion rates measuring the number of units. The reswitching puzzle is explained by redefining the independent variable to include simultaneously determined entities previously thought lacking in use and meaning.

Issues remain. First, the analysis is within the framework of comparative statics, as it was in the 1966 QJE symposium.

'Following Joan Robinson's strictures that it is most important not to apply theorems obtained from the analysis of differences to situations of change ... modern writers usually have been most careful to stress that their analysis is essentially the comparisons of different equilibrium situations one with another and that they are not analyzing actual processes.' Harcourt (1972, p. 122.)

The behavior described by Harcourt applies here. Different capital values are determined at different interest rates and compared, one with another, without reference to the passage of time. The incorporation of time into the analysis remains a challenge.

Second, there is more to the Cambridge capital controversies than the reswitching puzzle. What are the implications of the current analysis, if any, for the capital controversies overall?

Third, Sraffa's famous work, *Production of Commodities by Means of Commodities* (Sraffa, 1960), is subtitled *Prelude to a Critique of Economic Theory*. It is unlikely the implications of multiple-interest-rate analysis for economics and finance are what Sraffa had in mind by way of critique. Nevertheless, this interpretation of reswitching does have implications because the TVM equation is ubiquitous in economics and finance. Some topics have contemporary relevance, for example, Osborne (2010) employs multiple-interest-rate analysis to shed light on the longstanding NPV-IRR debate in investment appraisal and Osborne (2013) raises questions about the suitability of the annual percentage rate (APR) as a policy variable in consumer credit legislation. Many such topics remain open to exploration.

Finally, although the analysis attributes economic meaning to all unorthodox rates as a cluster, the meaning of a lone, unorthodox rate remains a puzzle.

Analysis of these issues is left for future research; in the meantime, this article provides a different perspective on a famous debate.

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Appendix A(i). The n^{th} order equation for the value of capital

Given a compounding equation of order n , like Eq. (A1), there are n solutions for the rate of interest r solving the equation.

$$C = L_1(1+r) + L_2(1+r)^2 + \dots + L_{n-1}(1+r)^{n-1} + L_n(1+r)^n \quad (\text{A1})$$

This fact implies the existence of n versions of Eq. (A1_j) from $j = 1$ to n , each equation having its own rate of interest r_j . The n equations hold true simultaneously.

$$C = L_1(1+r_1) + L_2(1+r_1)^2 + \dots + L_{n-1}(1+r_1)^{n-1} + L_n(1+r_1)^n \quad (\text{A1}_1)$$

$$C = L_1(1+r_2) + L_2(1+r_2)^2 + \dots + L_{n-1}(1+r_2)^{n-1} + L_n(1+r_2)^n \quad (\text{A1}_2)$$

...

$$C = L_1(1+r_{n-1}) + L_2(1+r_{n-1})^2 + \dots + L_{n-1}(1+r_{n-1})^{n-1} + L_n(1+r_{n-1})^n \quad (\text{A1}_{n-1})$$

$$C = L_1(1+r_n) + L_2(1+r_n)^2 + \dots + L_{n-1}(1+r_n)^{n-1} + L_n(1+r_n)^n \quad (\text{A1}_n)$$

Given Eq. (A1), and values for the labor inputs L_i , two possibilities exist. If C is known already, then all values of r_j are determined. Alternatively, if the orthodox rate of interest r_1 is known, then C and the remaining values of r_j are determined. When the order of the polynomial is high, the majority of the unorthodox solutions are complex-valued.

Appendix A(ii). The n^{th} order capital value equation is converted to its dual expression

The dual to the capital value equation (A1) in Appendix A(i) is derived as follows: Eq. (A1) is rearranged and factorized. The result is Eq. (A2) in which $[(1+r)-(1+r_j)]$ is the j^{th} factor, $(1+r_j)$ is the j^{th} root, and r_j is the interest rate implied by the j^{th} root.

$$\begin{aligned} L_n(1+r)^n + L_{n-1}(1+r)^{n-1} + \dots + L_2(1+r)^2 + L_1(1+r) - C = \\ L_n[(1+r)-(1+r_1)][(1+r)-(1+r_2)]\dots[(1+r)-(1+r_n)] \end{aligned} \quad (\text{A2})$$

As discussed in the text, the variable $(1+r)$ in Eq. (A2) can take any value, although some values are more interesting than others. In this instance, the salient value for $(1+r)$ is unity, i.e. r is zero. When r is zero, Eq. (A2) simplifies to Eq. (A3).

$$\sum_{i=1}^n L_i - C = L_n \prod_{j=1}^n (-r_j) \quad (\text{A3})$$

Eq. (A3) rearranges into Eq. (A4), which is the dual expression to the conventional capital value equation (A1).

$$C = \sum_{i=1}^n L_i - L_n \prod_{j=1}^n (-r_j) \quad (\text{A4})$$

Appendix B. The product of the unorthodox interest rates measures the number of times the orthodox interest rate is applied during amortization

In the text, the capital value equation (1a) is converted to the dual Eq. (13) in which X_r is the product of the two unorthodox interest rates r_2 and r_3 . When the unorthodox roots $(1+r_2)$ and $(1+r_3)$ are complex numbers, the implied interest rates take absolute values, i.e. the rates become $|r_2|$ and $|r_3|$, in which case $X_r = |r_2||r_3|$.

$$C = L_1(1+r_1) + L_2(1+r_1)^2 + L_3(1+r_1)^3 \quad (1a)$$

$$C = \sum_{i=1}^3 L_i + L_3 X_r r_1 \quad (13)$$

It is asserted in the text that X_r has meaning: the product of the unorthodox interest rates is equal to the number of times the orthodox interest rate r_1 is applied to an initially invested unit of labor during amortization of the conventional Eq. (1a). The amortization of Eq. (1a) is in Table B1. The analysis in the table reveals that the assertion is true if the following equality is true: $X_r = S/L_3$. This appendix contains a proof of the equality.

[Table B1 about here.]

Eq. (1a) is rearranged and written down four times in the form of a matrix. The first line of matrix M1 is Eq. (1a) divided throughout by $(1+r_1)$; the second line is Eq. (1a)

divided throughout by $(1+r_1)^2$; the third line is (1a) divided throughout by $(1+r_1)^3$; finally, the fourth line is (1a) divided throughout by $(1+r_1)^4$.

$$\begin{aligned}
0 &= -\frac{C}{(1+r_1)^1} + L_1(1+r_1)^0 + L_2(1+r_1)^1 + L_3(1+r_1)^2 \\
0 &= -\frac{C}{(1+r_1)^2} + \frac{L_1}{(1+r_1)^1} + L_2(1+r_1)^0 + L_3(1+r_1)^1 \\
0 &= -\frac{C}{(1+r_1)^3} + \frac{L_1}{(1+r_1)^2} + \frac{L_2}{(1+r_1)^1} + L_3(1+r_1)^0 \\
0 &= -\frac{C}{(1+r_1)^4} + \frac{L_1}{(1+r_1)^3} + \frac{L_2}{(1+r_1)^2} + \frac{L_3}{(1+r_1)^1}
\end{aligned} \tag{M1}$$

Every line in matrix (M1) sums to zero, therefore the matrix itself sums to zero. The six elements forming the triangle at the top right-hand corner of matrix (M1) comprise S in Table B1. It follows that the ten elements in the opposing triangle comprise -S; therefore Eq. (B1) is true.

$$-S = -C \left[\sum_{i=1}^4 \frac{1}{(1+r_1)^i} \right] + L_1 \left[\sum_{i=1}^3 \frac{1}{(1+r_1)^i} \right] + L_2 \left[\sum_{i=1}^2 \frac{1}{(1+r_1)^i} \right] + L_3 \left[\frac{1}{(1+r_1)} \right] \tag{B1}$$

The elements in square brackets in Eq. (B1) are simplified and the equation rearranged into Eq. (B2).

$$S = \frac{C}{r_1} \left[1 - \frac{1}{(1+r_1)^4} \right] - \frac{L_1}{r_1} \left[1 - \frac{1}{(1+r_1)^3} \right] - \frac{L_2}{r_1} \left[1 - \frac{1}{(1+r_1)^2} \right] - \frac{L_3}{r_1} \left[1 - \frac{1}{(1+r_1)} \right] \tag{B2}$$

Eq. (B2) is expanded and further rearranged into Eq. (B3).

$$Sr_1 = C - \sum_{i=1}^3 L_i + \left[-\frac{C}{(1+r_1)^4} + \frac{L_1}{(1+r_1)^3} + \frac{L_2}{(1+r_1)^2} + \frac{L_3}{(1+r_1)} \right] \tag{B3}$$

The element in square brackets on the right-hand side of Eq. (B3) is the bottom line of matrix (M1), which is zero. Eq. (B4) follows.

$$C = \sum_{i=1}^3 L_i + S r_l \quad (\text{B4})$$

Eq. (B4) is juxtaposed with Eq. (13).

$$C = \sum_{i=1}^3 L_i + L_3 X_r r_l \quad (13)$$

Comparing terms in equations (B4) and (13) proves the equality $X_r = S/L_3$. It follows that the product of the two unorthodox interest rates is equal to the number of times the orthodox rate r_l is applied to an initially invested unit of labor.

This proof generalizes, i.e. the proof can be applied to equations (A1) and (A4) in Appendices A(i) and A(ii).

In the text it is noted that Eq. (13) acts like a prism, splitting capital value (C) into the contribution from the uncompounded inputs (sum of L_i) and the contribution from the compounding process ($L_3 X_r r_l$). This attribute of the dual equation is emphasized here: Eq. (B4) splits capital value (C) into the contribution from the uncompounded labor inputs (sum of L_i) and the contribution from compounding, i.e. the total number of mark-ups performed during amortization (S) multiplied by the mark-up itself (r_l).

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Endnotes

1. There is a dispute in the literature about priority in perceiving reswitching. Cohen and Harcourt (2003) report the consensus view: ‘*As early as 1936, Sraffa wrote a letter to Robinson explaining the essence of this complication for neoclassical capital theory. Reswitching and capital-reversing were noted in the 1950s by David Champernowne ... and Robinson, but their full significance was realized only with Sraffa's 1960 book.*’ Velupillai (1975) notes that Fisher (1907, pp. 352-353) contains an example of reswitching but acknowledges that Fisher did not recognize the implications of the phenomenon as Sraffa and his colleagues did; therefore the consensus prevails.
2. The words ‘profit’ and ‘interest’ are used interchangeably in the reswitching literature. To avoid confusion and repetition, from this point the word ‘interest’ is employed.
3. Software written for mathematicians, scientists and engineers is not so restricted. For example, entering the formula $\text{solve}(9.262=6x+2x^3,x)$ into www.wolframalpha.com yields all three values of $x=(1+r)$ solving Eq. (3) when C_B is 9.262.
4. The elements in parentheses in Eq. (11) require comment. When the j^{th} root $(1+r_j)$ is a real number, the factor $[(1+r)-(1+r_j)]$ is $(-r_j)$ when $r = 0$. Real solutions give rise to one of two outcomes. When the intrinsic value of r_j is positive then the j^{th} element in parentheses is a negative number because of the negative sign inside the parentheses. When the intrinsic value of r_j is negative then the j^{th} element in parentheses is a positive number because the negative signs inside the parentheses negate each other.

When the j^{th} root $(1+r_j)$ is a complex number, the absolute value of the factor is taken, i.e. $[(1+r)-(1+r_j)]$ becomes $|r_j|$ when $r = 0$. In this situation the j^{th} element in parentheses is a positive, real number $|r_j|$ representing a distance in the complex plane between the root $(1+r_j)$ and the point $(1,0)$.
5. ‘*A mathematical variable x is ‘something’ or, more accurately, ‘anything’ that may take on various numerical values.*’ Aleksandrov *et al.* (1969)

Table 1. The Sraffa-Samuelson example: labor coefficients for two techniques, A & B, over three time periods

Time period	Labor inputs for technique A	Labor inputs for technique B
1 (last period)	$L_1 = 0$	$L_1 = 6$
2 (two periods ago)	$L_2 = 7$	$L_2 = 0$
3 (three periods ago)	$L_3 = 0$	$L_3 = 2$

Table 2a. The rates of interest solving the second-order TVM polynomial resulting from the labor input {0, 7, 0} in technique A

Two equations for capital value with identical labor inputs, each equation having its own solution	The values of $(1+r_{Aj})$ given $r_{A1} = 10\%$.	The implied values of $(-r_{Aj})$
$8.47 = 7(1+r_{A1})^2$	$(1+r_{A1}) = 1.1$	$(-r_{A1}) = -0.1$
$8.47 = 7(1+r_{A2})^2$	$(1+r_{A2}) = -1.1$	$(-r_{A2}) = 2.1$

Table 2b. The rates of interest solving the third-order TVM polynomial resulting from the labor inputs {6, 0, 2} in technique B

Three equations for capital value with identical labor inputs, each equation having its own solution	The values of $(1+r_{Bj})$ given $r_{B1} = 0.1 = 10\%$.	The implied values of $(-r_{Bj})$
$9.262 = 6(1+r_{B1}) + 2(1+r_{B1})^3$	$(1+r_{B1}) = 1.1$	$(-r_{B1}) = -0.1$
$9.262 = 6(1+r_{B2}) + 2(1+r_{B2})^3$	$(1+r_{B2}) = -0.55 + 0.97674i$	$ r_{B2} = 2.51197$
$9.262 = 6(1+r_{B3}) + 2(1+r_{B3})^3$	$(1+r_{B3}) = -0.55 - 0.97674i$	$ r_{B3} = 2.51197$

Table 3. The amortization schedule for technique A

Col.1	Col.2	Col.3	Col.4
Time	Labor inputs	Units of labor outstanding	Number of labor units marked up at each moment in time at the rate $(1+r_{AI}) = (1.1)$
2	7	7	
1	0	$7(1.1) + 0$	7
0	-8.47	$7(1.1)^2 + 0(1.1) - 8.47$	$7(1.1) + 0$
		The last equation above is Eq. (4) and it is equal to zero.	Total number of labor units marked up at the rate $(1+r_{AI}) = (1.1)$ is the sum of the cells above, i.e. $S = 14.7$
Number of times an initially invested labor unit is marked up: $S/L_2 = 14.7 / 7 = 2.1$			
Number of times $r_{AI} = 0.1$ is applied during amortization is equal to the lone unorthodox mark-up $(-r_{A2}) = 2.1$.			

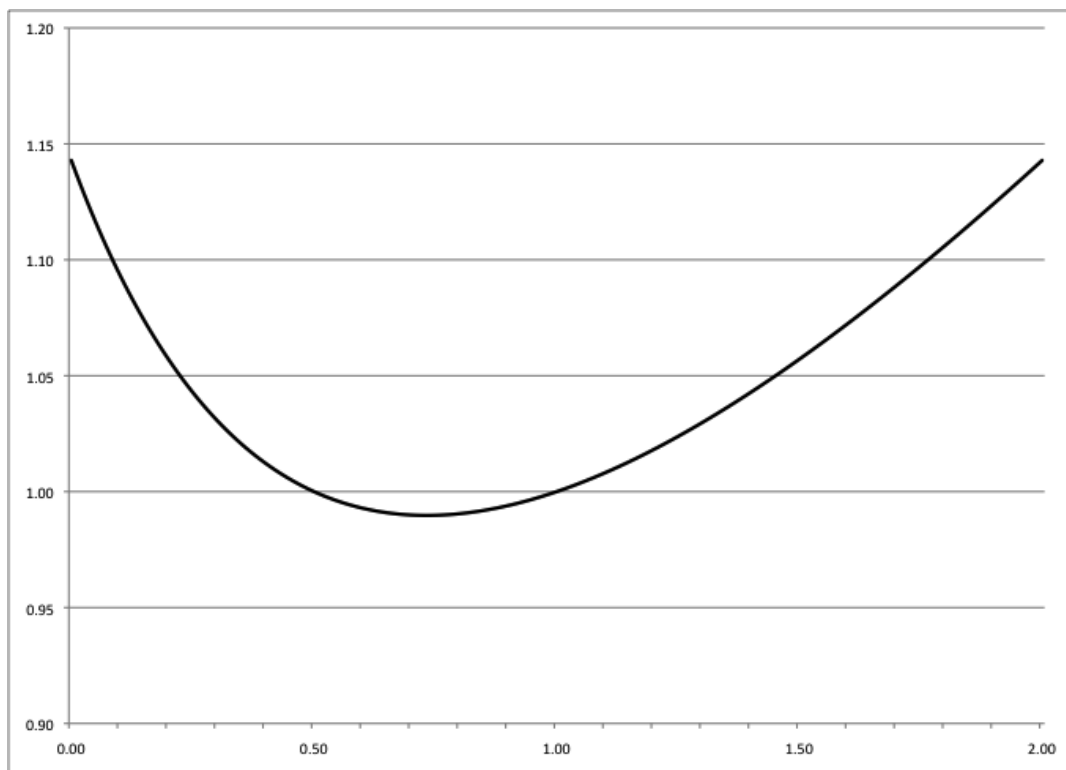
Table 4. The amortization schedule for technique B

Col.1	Col.2	Col.3	Col.4
Time	Labor inputs	Units of labor outstanding	Number of labor units marked up at each moment in time at the rate $(1+r_{BI}) = (1.1)$
3	2	2	
2	0	$2(1.1) + 0$	2
1	6	$2(1.1)^2 + 0(1.1) + 6$	$2(1.1) + 0$
0	-9.262	$2(1.1)^3 + 0(1.1)^2 + 6(1.1) - 9.262$	$2(1.1)^2 + 0(1.1) + 6$
		The last equation above is Eq. (6) and it is equal to zero.	Total number of labor units marked up at the rate $(1+r_{BI}) = (1.1)$ is the sum of the cells above, i.e. $S = 12.62$
Number of times an initially invested labor unit is marked up: $S/L_3 = 12.62 / 2 = 6.31$			
Number of times $r_{BI} = 0.1$ is applied during amortization is equal to the product of the two unorthodox mark-ups $ r_{B2} $ and $ r_{B3} $, i.e. $2.51197 \times 2.51197 = 6.31$.			

Table B1. Amortization schedule for the three-period capital value function, Eq. (1a)

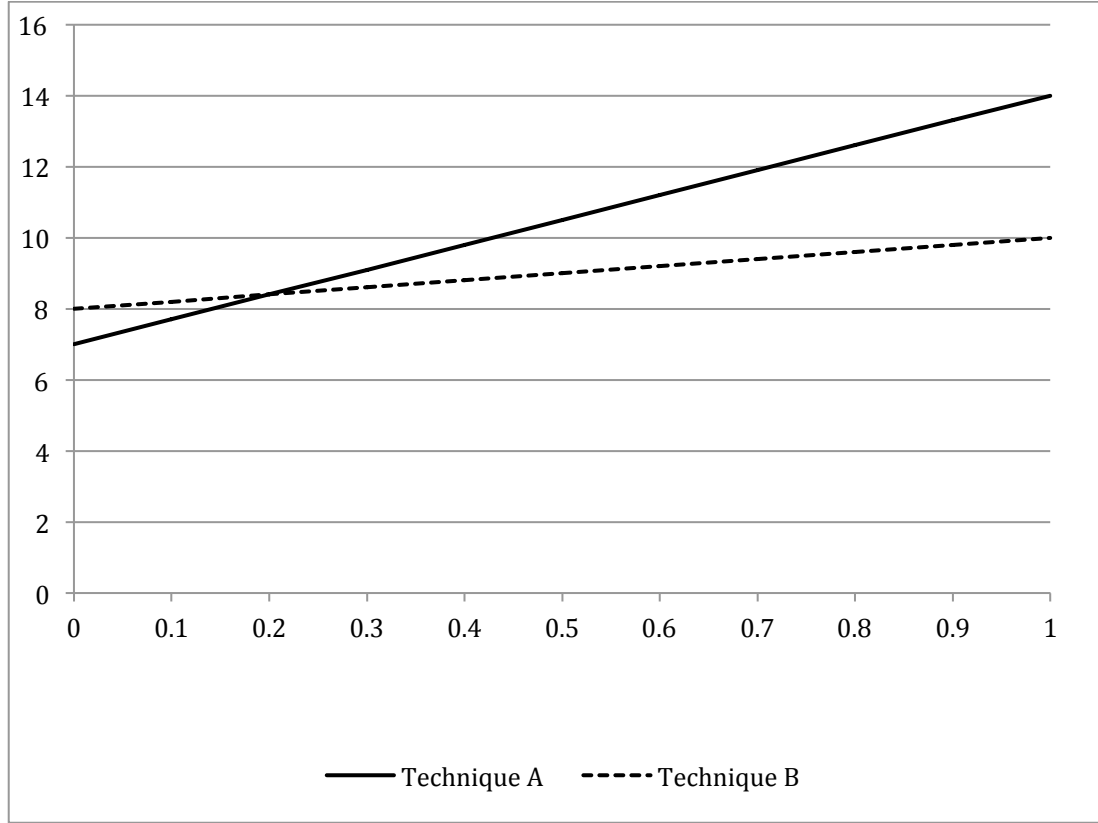
Col.1	Col.2	Col.3	Col.4
Time	Labor inputs	Cumulative units of labor	The number of labor units marked up at each moment in time at the rate $(1+r_I)$
3	L_3	L_3	
2	L_2	$L_3(1+r_I) + L_2$	L_3
1	L_1	$L_3(1+r_I)^2 + L_2(1+r_I) + L_1$	$L_3(1+r_I) + L_2$
0	$-C$	$L_3(1+r_I)^3 + L_2(1+r_I)^2 + L_1(1+r_I) - C$	$L_3(1+r_I)^2 + L_2(1+r_I) + L_1$
		The last equation above is equal to zero.	The total number of labor units marked up at the rate $(1+r_I)$ = the sum of cells above = S
The number of times an initially invested labor unit is marked up is S/L_3 .			

Figure 1. Sraffa-Samuelson model: ratio of capital values (technique B to technique A) on the y-axis is a function of rate of interest on the x-axis.



Switching and reswitching take place when the orthodox interest rate is equal to 50% and 100% respectively.

Figure 2. Capital values for technique A and technique B on the y-axis vary linearly with the composite variable X, r_I on the x-axis



There is switching but no reswitching. Switching takes place when $C_A = C_B = 8.4$. When $C_A = 8.4$, the interest rates r_{A1} and r_{A2} are equal to 0.095445 and -2.095445 respectively. When $C_B = 8.4$, the interest rates r_{B1} , $|r_{B2}|$ and $|r_{B3}|$ are equal to 0.03279, 2.46971 and 2.46971 respectively. At the switch point, it is not the orthodox interest rates r_{A1} and r_{B1} that are equal; instead, it is the products of interest rates that are equal, as depicted in Eq. (16) and Eq. (17), i.e. $-(-r_{A1}) * (-r_{A2}) = -(-r_{B1}) * |r_{B2}| * |r_{B3}| = 0.2$. All interest rates are simultaneously determined, equally valid, and *jointly* convey meaning as the independent variable: their product is a ‘global’ mark-up applied to the parameters (the given labor inputs). Capital value increases linearly with the ‘global’ mark-up, but at different rates for the two different techniques.